Exercise 1. Let $F : X \to \mathbb{R} \cup \{\infty\}$ be a coercive and sequentially lower semi-continuous function. Assume that $K \subset X$ is a sequentially closed set and define

$$F_K(x) = \begin{cases} F(x) & \text{for all } x \in K \\ \infty & \text{otherwise.} \end{cases}$$

Prove that F_K is coercive and sequentially lower semi-continuous.

Exercise 2 (Lemma of Du Bois-Reymond)**.** Let $f \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} f(x)\varphi'(x)dx = 0 \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}). \tag{1}$$

Prove that there exists $C \in \mathbb{R}$ such that $f = C$ identically.

Hint : Consider the hyperplane

$$H = C_c^\infty(\mathbb{R}) \cap \left\{\varphi : \int_{\mathbb{R}} \varphi(x)dx = 0\right\}.$$

What is the value of (1) for $\varphi \in H$?

Exercise 3 (Approximation of Lipschitz functions)**.** Let $u : [-1,1] \to \mathbb{R}, t \mapsto |t| - 1$. Show that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathscr{D}(]-1,1[)$ such that $u_k \underset{k\to\infty}{\longrightarrow} u$ uniformly on $[-1,1]$, and $u_k' \underset{k\to\infty}{\longrightarrow} u$ pointwise on $]1,0[\cup]0,1[$ and that $\{u_k'\}_{k\in\mathbb{N}}$ is bounded.

Exercise 4 (Euler-Lagrange equation)**.** Let $F \in C^1(\mathbb{R}^3, \mathbb{R})$ and define the Lagrangian

$$L(u) = \int_{-1}^{1} F(t, u(t), u'(t))dt,$$

and let $\alpha, \beta \in \mathbb{R}$. We look at functions $u : [-1,1] \to \mathbb{R}$ satisfying the following constraints :

$$\begin{cases} u \in C^1([a,b]) \\ u(-1) = \alpha \\ u(1) = \beta. \end{cases}$$

1. If $F(t,x,p) = p^2 - p^4$ and $\alpha = \beta = 0$, find all solutions of the Euler-Lagrange equation under the above constraints. Does there exists a minimiser ?
2. If $F(t,x,p) = x^2(2t-p)^2$ and $\alpha = 0$ and $\beta = 1$, find all solutions of the Euler-Lagrange equation under the above constraints. Does there exists a minimiser ? If so, is it unique ?